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MORRIS-THORNE TRAVERSABLE WORMHOLE WITH A GENERIC COSMOLOGICAL CONSTANT

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Abstract

The static and spherically symmetric Morris-Thorne traversable wormhole solutions in the presence of cosmological constant are analyzed. We matched an interior solution of a spherically symmetric traversable wormhole to a unique exterior vacuum solution at a junction surface. The surface tangential pressure on the thin layer of shell is deduced. The specific wormhole solutions are constructed with generic cosmological constant.

I INTRODUCTION

Wormholes are handles or tunnels in the spacetime topology connecting two separate and distinct regions of spacetime. These regions may be part of our Universe or of different Universes. The static and spherically symmetric traversable wormhole was first introduced by Morris and Thorne in their classic paper [1]. From the stand point of cosmology,the cosmological constant Λ, served to create a kind of repulsive pressure to yield a stationary Universe. Zel'dovich [2] identified Λ with the vacuum energy density due to quantum fluctuations. Morris-Thorne wormholes with a cosmological constant Λ have been studied extensively, even allowing Λ to be replaced by a space variable scalar field. These wormholes cannot exist, however, if Λ are both space and time dependent. Such a Λ will therefore act as a topological censor.

In this article, we introduce an exact black hole solution of the Einstein field equations in four dimensions with a positive cosmological constant to electromagnetic and conformally coupled scalar fields. This solution is often called a Martinez-Troncoso-Zanelli (MTZ) black hole solution. In agreement with recent observations [3], this black hole only exists for a positive cosmological constant Λ, and if a quartic self-interaction coupling is considered. Static scalar field configurations such as those presented here, which are regular both at the horizon as well as outside, are unexpected in view of the no-hair conjecture [4]. The conformal coupling for the scalar field is the unique prescription that guarantees the validity of the equivalence principle in curved spacetime [5]. In the literature, a number of traversable wormhole solutions with cosmological constant are available [6-21]. A general class of wormhole geometries with a cosmological constant and junction conditions was analyzed by De Benedictis and Das [9], and further explored in higher dimensions [10]. It is of interest to study a positive cosmological constant, as the inflationary phase of the ultra-early universe demands it, and in addition, recent astronomical observations point to $\Lambda > 0$. Lobo [12], with the intension of minimizing the exotic matter used, matched a static and spherically symmetric wormhole solution to an exterior vacuum solution with a cosmological constant, and he calculate the surface stresses of the resulting shell and the total amount of exotic matter using a volume integral quantifier [13]. The construction of traversable wormhole solutions by matching an interior wormhole spacetime to an exterior solution, at a junction surface, was analyzed in [13-15]. A thin-shell traversable wormhole, with a zero surface energy density was analyzed in [15], and with generic surface stresses in [14]. A general class of wormhole geometries with a cosmological constant and junction conditions was explored in [9], and a linearized stability analysis for the plane symmetric case with a negative cosmological constant is done in [17].

Morris-Thorne wormholes, with $\Lambda = 0$, have two asymptotically flat regions spacetime. By adding a positive cosmological constant $\Lambda > 0$, the wormholes have two asymptotically de-Sitter regions, and by adding a negative cosmological constant, $\Lambda < 0$, the wormholes have two asymptotically anti-de Sitter regions. We analyze asymptotically flat and static traversable Morris-Thorne wormholes in the presence of a cosmological construct. An equation connecting the radial tension at the mouth with the tangential surface pressure of the thin-shell is derived. The structure as well as several physical properties and characteristics of traversable wormholes due to the effects of the cosmological term are studied.

This article is organized as follows: In Sec. II we studied Einstein's field equations and total stress-energy with a cosmological constant Λ. In Sec. III, we introduce an exact black hole solution with electromagnetic and conformally coupled scalar fields. The junction conditions and the surface tangential pressure are discussed in Sec.

IV. Specific construction of wormhole with generic cosmological constant is discussed in Sec. V. Finally, conclusion of the results is given in Sec. VI.

II EINSTEIN'S FIELD EQUATIONS AND SURFACE STRESSES WITH A COSMOLOGICAL CONSTANT Λ

a) Form of the Spacetime Metric

The interior spacetime metric for the wormhole in the static and spherically symmetric isotropic coordinate (t, r, θ, ϕ), is given by [1]

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{\left(1-\frac{b(r)}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $\Phi(r)$ is denoted as the redshift function, for it is related to the gravitational redshift and $b(r)$ is called the form function, as it determines the shape of the wormhole; both are functions of the radial coordinate. For the traversable wormhole, one must demand that there are no horizons present, which are identified as the surfaces with $e^{-2\Phi} \to 0$, so the $\Phi(r)$ must be finite everywhere. The radial coordinate has a range that increases from a minimum value at r_0, corresponding to the wormhole throat to a. Maximum value of a corresponding to the mouth at r_0 one has to join smoothly this spherical volume to another one copy with r ranging again from r_0 to a. In addition, one has then to join each copy to the external spacetime from a to ∞, as will be done.

The details of subsequent mathematics and of physical interpretations will be simplified using a set of orthonormal basis vectors as the proper reference frame, the observers remain at rest in this coordinate system (t, r, θ, ϕ), with (r, θ, ϕ) constant. The basis vectors in this coordinate system are denoted by $e_t, e_r, e_\theta, e_\phi$. The transformation of these basis vectors from the proper reference frame to a boosted frame is as follows:

$$e_t = e^{-\phi}e_t, \qquad e_{\hat{r}} = (1-b/r)^{1/2}e_r$$
$$e_{\hat{\theta}} = r^{-1}e_\theta \quad \text{and} \quad e_{\hat{\phi}} = (r\sin\theta)^{-1}e_\phi. \quad (2)$$

In this basis the metric coefficients assume on their standard, special relativity forms are given by

$$g_{\hat{\mu}\hat{\nu}} = e_{\hat{\mu}}.e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} = diag(-1,1,1,1). \quad (3)$$

In the orthonormal reference frame, the Einstein field equation with a generic cosmological constant can be written as

$$G_{\hat{\mu}\hat{\nu}} + \Lambda\eta_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}}. \quad (4)$$

b) The Total Stress-Energy Tensor with a Cosmological Constant

One may write the Einstein field equation with a cosmological constant in the following manner;

$$G_{\hat{\mu}\hat{\nu}} = 8\pi G\left(T_{\hat{\mu}\hat{\nu}} + T_{\hat{\mu}\hat{\nu}}^{(vac)}\right), \quad (5)$$

where $T_{\hat{\mu}\hat{\nu}}^{(vac)} = -g_{\hat{\mu}\hat{\nu}}\left(\Lambda/(8\pi G)\right)$, is the stress-energy tensor associates with the vacuum, and in the orthonormal reference frame is given by

$$T_{\hat{\mu}\hat{\nu}}^{(vac)} = diag\left[\Lambda/(8\pi G) - \Lambda/(8\pi G) - \Lambda/(8\pi G)\Lambda/(8\pi G)\right]. \quad (6)$$

For the metric (1), the non-zero components of the Einstein tensor in the orthonormal reference frame can be written as [1]

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}, \quad (7)$$

$$G_{\hat{r}r} = -\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \quad (8)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \left(1-\frac{b}{r}\right)\left[\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right]. \quad (9)$$

Using the Einstein field equations with a non-zero cosmological constant in an orthonormal reference frame, we obtain the following stress-energy scenario

$$\tau(r) = \frac{1}{8\pi G}\left[\frac{b}{r^3} - 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r} - \Lambda\right], \quad (11)$$

$$p(r) = \frac{1}{8\pi G}\left\{\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi' - \frac{b'r-b}{2r^2(r-b)} + \frac{\Phi'}{r}\right] + \Lambda\right\}, \quad (12)$$

where $\rho(r)$ is the energy density, $\tau(r)$ is the radial tension, $p(r)$ is the pressure measured in the lateral directions, orthonormal to the radial direction.

We obtain the equation for τ' by taking the derivative of Eq. (11) with respect to the radial coordinate **r** and eliminating b' and Φ'', given in Eqs. (10) and (12), respectively,

$$\tau' = \left(\rho c^2 - \tau\right)\Phi' - \frac{2}{r}(\rho + \tau). \quad (13)$$

Equation (13) is known as the relativistic Euler equation or the hydrostatic equilibrium equation for the material threading the wormhole. This equation can also be obtained using the conservation of the stress-energy tensor $T^{\hat{\mu}\hat{\nu}}_{;} = 0$, putting $\mu' = r$. The conservation of the stress-energy tensor can also be deduced from the Bianchi identities, which are equivalent to $G^{\hat{\mu}\hat{\nu}}_{;\hat{\nu}} = 0$.

III EXTERIOR SOLUTION WITH GENERIC Λ_{ext}

The exterior vacuum solution of Einstein field equations is given by

$$ds^2 = -\left[-\frac{\Lambda_{ext}}{3}r^2 + \left(1-\frac{GM}{r}\right)^2\right]dt^2 + \left[-\frac{\Lambda_{ext}}{3}r^2 + \left(1-\frac{GM}{r}\right)^2\right]^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (14)$$

where $0 \le r < \infty$. This is the solution of de-Sitter black hole with a conformally coupled scalar field and also known as MTZ solution. The scalar field is given by

$$\phi(r) = \sqrt{\frac{3}{4\pi}}\frac{\sqrt{G}M}{r-GM}. \quad (15)$$

The MTZ solution exists only for a dimensionless constant, $\alpha = -\frac{2}{9}\pi\Lambda G$, and describe a static and spherically symmetric black hole with a positive cosmological constant Λ. The mass of the black hole satisfies $0 > GM > l/4$, where l is the cosmological radius and is given by $l = \sqrt{3/\Lambda_{ext}}$. The inner, event and cosmological horizon satisfies $0 < r_- < GM < r_+ < l/2 < r_c < l$, where

$$r_- = \frac{l}{2}\left(-1 + \sqrt{1 + 4GM/l}\right), \quad (16)$$

$$r_+ = \frac{l}{2}\left(1 - \sqrt{1 - 4GM/l}\right), \quad (17)$$

$$r_{++} = \frac{l}{2}\left(1 + \sqrt{1 - 4GM/l}\right). \quad (18)$$

The solution (14) have singularities at the radii $r_{\pm} = \frac{l}{2}\left(\pm 1 \mp \sqrt{1-4GM/l}\right)$. $r = r_+ = r_b$ can be considered as the event horizon of the vacuum black hole solution, but since the wormhole matter will fill region up to a wormhole radius a superior than r_b. This radius does not enter into the problem. For the same reason, $r = r_-$, the inner event horizon of the black hole is not considered in the present problem. So $r = r_{++} = r_c$ can be regarded as the position of the cosmological event horizon of the de-Sitter spacetime. Keeping Λ_{ext} fixed, if one increases M, $r = r_+$ will increase and $r = r_{++}$ will decrease. For the maximum allowed value of the mass, $M = l(4G)^{-1}$, the black hole event horizon and cosmological horizon are same i.e., $r_+ = r_{++} = l/2$. In the case of vanishing cosmological constant $\Lambda_{ext} = 0$, the geometry of the extreme Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{19}$$

which has coalesced inner and event horizons at $r_+ = r_- = GM$. For the massless case, $M = 0$, the black hole geometry in de-Sitter spacetime and the metric takes a simple form

$$ds^2 = -\left(1 - \frac{\Lambda_{ext}}{3} r^2\right) dt^2 + \frac{dr^2}{\left(1 - \frac{\Lambda_{ext}}{3} r^2\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{20}$$

For $\Lambda_{ext} \to 0$, the de-Sitter metric tends to the Minkowskian spacetime.

IV JUNCTION CONDITIONS

In order to match the interior and exterior matrices, one needs the boundary surface S that connects them. The first condition is that the metric must be continuous at S, i.e., $g_{\mu\nu}^{\text{int}}\big|_S = g_{\mu\nu}^{ext}\big|_S$. This condition is not sufficient to join different spacetimes. The second condition for making the match can be done directly with the field equation, due to the spherically symmetric. We can use the Einstein field equations, Eqs. (7), (8) and (9), to determine the energy density and stresses of the surface necessary to have a match between the interior and exterior solutions. When there is null stress-energy terms at S, we can say that the junction is a boundary surface. On the other hand, if surface stress-energy terms are present, the junction is called the thin-shell.

Since both the inside and outside matrices are spherically symmetric, the components $G_{\theta\theta}$ and $G_{\phi\phi}$ are already continuous, and therefore one is left with imposing the continuity G_{tt} and G_{rr}, these can be written as $g_{tt}^{\text{int}}\big|_{r=a} = g_{tt}^{ext}\big|_{r=a}$ and $g_{rr}^{\text{int}}\big|_{r=a} = g_{rr}^{ext}\big|_{r=a}$. At $r = a$, with g_{tt}^{int} and g_{rr}^{int} being the metric components for the interior region at $r = a$, and g_{tt}^{ext} and g_{rr}^{ext} the exterior metric components for the vacuum solution at $r = a$. We are considering the interior solution Eq. (1) and the MTZ exterior solution Eq. (14) matched at surface, S. The continuity of the matrices then give generically $\Phi_{\text{int}}(a) = \Phi_{ext}(a)$ and $b_{\text{int}}(a) = b_{ext}(a)$. Now comparing Eqs. (1) and (14), the red shift and shape functions can be written as

$$\Phi(a) = \frac{1}{2}\ln\left[-\frac{\Lambda_{ext}}{3}a^2 + \left(1 - \frac{GM}{a}\right)^2\right], \tag{21}$$

$$b(a) = 2GM - \frac{G^2M^2}{a} + \frac{\Lambda_{ext}}{3}a^3. \tag{22}$$

We consider a particular choice in which the static interior observer measures zero tidal forces, i.e., $\Phi_{\text{int}} = const.$ and $\Phi'_{\text{int}} = 0$. Since the shell is infinitesimally thin in the radial direction, so there is no radial surface pressure. Therefore we are left with a surface energy density σ and a surface tangential pressure P.

At the boundary S, the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$ is proportional to a Dirac delta function, so one can write $T_{\hat{\mu}\hat{\nu}} = t_{\hat{\mu}\hat{\nu}}\,\delta(\hat{r} - \hat{a})$. To find $t_{\hat{\mu}\hat{\nu}}$ we then use

$$\int_{-}^{+} G_{\hat{\mu}\hat{\nu}} d\hat{r} = 8\pi G \int_{-}^{+} t_{\hat{\mu}\hat{\nu}} \delta(\hat{r} - \hat{a}) d\hat{r}, \tag{23}$$

where $\int_{-}^{+}$ means an infinitesimal integral through the shell. Now using the property of the δ function $\delta(f(x)) = [1/|f'(x)|]\delta(x)$, and $\int_{-}^{+} g(x)\delta(x - x_0) = g(x_0)$, we find

$$t_{\hat{\mu}\hat{\nu}} = \frac{1}{8\pi G} \int_{-}^{+} G_{\hat{\mu}\hat{\nu}} d\hat{r}. \tag{24}$$

We see that $G_{\hat{t}\hat{t}} = \frac{b'}{r^2}$ only depends on the first derivative of the metric, which are continuous for interior and exterior solutions. Thus, since the integral gives the value of the metric on the exterior side (b^+, say) minus the value of the metric on the interior side (b^-, say), it gives zero, and one finds $\sigma = 0$.

In Eq. (9) we see that $G_{\hat{\theta}\hat{\theta}}$ has an important term $[1 - (b/r)]\Phi''$, the other terms in this equation depend at most on the first derivative and do not contribute to the integral. From Eq. (23), we obtain $8\pi G/p = \sqrt{1 - b(a)/a}\, \Phi_{-}^{+}$. Now $\Phi_{-}^{+} = 0$ is taken before and $\Phi^{+} = \left(\frac{GM}{a^2} - \frac{G^2 M^2}{a^3} - \frac{\Lambda_{ext}}{3} a \right) \Big/ (1 - b(a)/a)$. Therefore, the surface tangential pressure can be obtained as

$$P = \frac{1}{8\pi Ga} \frac{\frac{GM}{a} - \frac{G^2 M^2}{a^2} - \frac{\Lambda_{ext}}{3} a^2}{\sqrt{1 - b(a)/a}}. \tag{25}$$

This equation can be written more explicitly as

$$P = \frac{1}{8\pi Ga} \frac{\frac{GM}{a} - \frac{G^2 M^2}{a^2} - \frac{\Lambda_{ext}}{3} a^2}{\sqrt{\left(1 - \frac{GM}{a}\right)^2 - \frac{\Lambda_{ext}}{3} r^2}}. \tag{26}$$

One can obtain the matching equation of the radial pressure across the junction boundary of the thin-shell. This is done by considering two general solutions of Eq. (1), and an interior and exterior solutions matched at the junction surface. The radial component of the Einstein Eq. (11), provides us

$$\frac{b_{\text{int}}}{r^3} = 8\pi G \;\; \tau_{\text{int}}(r) + \Lambda_{\text{int}} + 2\left(1 - \frac{b_{\text{int}}}{r}\right) \frac{\Phi'_{\text{int}}}{r}, \tag{27}$$

$$\frac{b_{ext}}{r^3} = 8\pi G \;\; \tau_{ext}(r) + \Lambda_{ext} + 2\left(1 - \frac{b_{ext}}{r}\right) \frac{\Phi'_{ext}}{r}. \tag{28}$$

At the junction boundary, one has obtained $\Phi_{\text{int}}(a) = \Phi_{ext}(a)$ and $b_{\text{int}}(a) = b_{ext}(a)$. For simplicity, we are considering $\Phi'_{\text{int}}(a) = 0$. From Eq. (21) we have

$$\Phi'(r)=\frac{\left(\frac{GM}{r^2}-\frac{G^2M^2}{r^3}-\frac{\Lambda_{ext}}{3}r\right)}{\left[\left(1-\frac{GM}{r}\right)^2-\frac{\Lambda_{ext}}{3}r^2\right]}. \tag{29}$$

Using Eqs. (29) and (26), we verify that Eqs. (27) and (28) provide us with an equation which governs the behavior of the radial tension at the boundary, namely

$$\tau_{\text{int}}(a)+\frac{1}{8\pi G}\Lambda_{\text{int}}=\tau_{ext}(a)+\frac{1}{8\pi G}\Lambda_{ext}+\frac{2}{a}Pe^{\Phi(a)}, \tag{30}$$

where we have put $e^{\Phi(a)}=\sqrt{\left(1-\frac{GM}{a}\right)^2-\frac{\Lambda_{ext}}{3}a^2}$. This equation relates the radial tension at the surface with the tangential pressure of the thin-shell.

V SPECIFIC CONSTRUCTION OF WORMHOLE WITH GENERIC Λ

To construct a specific wormhole solutions with generic cosmological constant Λ, we briefly discuss the two cases $\Lambda_{ext}=0$, $\Lambda_{ext}>0$. The specific wormhole solutions are given below.

a) Specific Traversable Wormhole Solution with $\Lambda_{ext}=0$

a.1) Junction with $P=0$

With the junction having the tangential pressure, $P=0$, we consider a matching of an interior solution to an exterior MTZ solution, so we have $\tau_{ext}=0$ and $\Lambda_{ext}=3/l^2$. In the case of the boundary surface, i.e. $P=0$, we obtain $\Lambda_{ext}=0$. Thus there is no wormhole solution with $P=0$.

a.2) Junction with $P\neq 0$

Again we consider a matching of the interior solution to an exterior MTZ solution with the tangential pressure of the junction, $P\neq 0$, we have $\tau_{ext}=0$ and $\Lambda_{ext}=0$. At the junction of the shell, the behavior of the radial tension is given by Eq. (30) and considering Eq. (22) we find the shape function at the junction simply reduces to $b(a)=2GM-\frac{G^2M^2}{a}$. For different wormhole solutions, we shall consider various choice of the shape function $b(r)$.

1. First we consider the wormhole solution for the functions

$$b(r)=(r_o r)^{1/2} \quad ; \quad \Phi(r)=\Phi_0. \tag{31}$$

where r_0 is the throat radius of the wormhole. The Einstein field equations are given by

$$\rho(r)+\frac{1}{8\pi G}\Lambda_{\text{int}}=\frac{1}{16\pi G}\frac{r_0^{1/2}}{r^{5/2}}, \tag{32}$$

$$\tau(r)+\frac{1}{8\pi G}\Lambda_{\text{int}}=\frac{1}{8\pi G}\frac{r_0^{1/2}}{r^{5/2}}, \tag{33}$$

$$p(r)+\frac{1}{8\pi G}\Lambda_{\text{int}}=\frac{1}{32\pi G}\frac{r_0^{1/2}}{r^{5/2}} \tag{34}$$

In this case the energy density ρ can be positive or zero, depending on the value of the internal cosmological constant Λ_{int}. The throat radius of the wormhole after matching the shape functions $b(a)=2GM-\frac{G^2M^2}{a}$ and

$b(r)=(r_0 r)^{1/2}$ must be greater than the black hole radius. The constant ϕ_0 must satisfy $e^{2\Phi_0}=\left(1-\frac{GM}{r}\right)^2$ and the rescaling interior metric, $r_0 \le r \le a$, is given by

$$ds^2 = -\left(1-\frac{GM}{a}\right)^2 dt^2 + \frac{dr^2}{\left(1-\sqrt{\frac{r_0}{r}}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (35)$$

while the exterior metric, $a \le r \le \infty$, is the MTZ solution (14).

2. Second specific wormhole solution is

$$b(r) = \frac{r_0^2}{r}; \qquad \Phi(r) = \Phi_0, \qquad (36)$$

where r_0 is the throat radius of the wormhole. The Einstein field equations are given by

$$\rho(r) + \frac{1}{8\pi G}\Lambda_{\text{int}} = -\frac{1}{8\pi G}\frac{r_0^2}{r^4}, \qquad (37)$$

$$\tau(r) + \frac{1}{8\pi G}\Lambda_{\text{int}} = \frac{1}{8\pi G}\frac{r_0^2}{r^4}, \qquad (38)$$

$$p(r) + \frac{1}{8\pi G}\Lambda_{\text{int}} = \frac{1}{32\pi G}\frac{r_0^2}{r^4}. \qquad (39)$$

In this case the energy density ρ can be positive or zero, depending on the value of the internal cosmological constant Λ_{int}.

The radius of the wormhole throat after matching the two shape functions $b(a)=2GM-\frac{G^2M^2}{a}$ and $b(r)=\frac{r_0^2}{r}$ must be greater than the black hole radius. The constant ϕ_0 must satisfy $e^{2\Phi_0}=\left(1-\frac{GM}{r}\right)^2$. To find the interior metric of the wormhole, we must impose the condition, $r_0 \le r \le a$, and this is given by

$$ds^2 = -\left(1-\frac{GM}{a}\right)^2 dt^2 + \frac{dr^2}{\left(1-\frac{r_0^2}{r^2}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (40)$$

The exterior metric, $a \le r \le \infty$, is the MTZ metric (14).

b) Specific Traversable Wormhole Solution with $\Lambda_{ext} > 0$

b.1) Junction with $P=0$

Now we shall consider the matching of an interior solution to an exterior MTZ solution, $\tau_{ext}=0$ and $\Lambda_{ext}>0$, at a boundary surface, $P=0$. One may obtain Eq. (30) that holds the following condition

$$\tau_{\text{int}}(a) + \frac{1}{8\pi G}\Lambda_{\text{int}} = \frac{1}{8\pi G}\Lambda_{ext}, \qquad (41)$$

at the boundary surface. Now in view of Eq. (41), we have $b(a)=\Lambda_{ext}a^3$. We shall consider identical shape functions as in the previous section.

1. First specific wormhole solution for the following functions:

$$b(r) = (r_0 r)^{1/2}; \qquad \Phi(r) = \Phi_0. \tag{42}$$

From matching the shape functions $b(a) = \Lambda_{ext} a^3$ and $b(r) = (r_0 r)^{1/2}$, one can find the radius of the wormhole and this radius must be greater than the black hole radius. Moreover, the constant ϕ_0 must satisfy the red shift function $e^{2\Phi_0} = -\frac{a^2}{l^2} + \left(1 - \frac{GM}{r}\right)^2$. The rescaling interior metric of the wormhole at $r_0 \le r \le a$, is given by

$$ds^2 = -\left[-\frac{a^2}{l^2} + \left(1 - \frac{GM}{r}\right)^2\right] dt^2 + \frac{dr^2}{\left(1 - \sqrt{\frac{r_0}{r}}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{43}$$

The rescaling exterior metric, $a \le r \le \infty$, is the same as exterior solution of MTZ black hole (14).

2. The second specific wormhole solution is

$$b(r) = \frac{r_0^2}{r}; \qquad \Phi(r) = \Phi_0. \tag{44}$$

The radius of the wormhole after matching the shape functions $b(a) = \Lambda_{ext} a^3$ and $b(r) = \frac{r_0^2}{r}$ must be greater than the black hole radius. In the case, the red shift function takes the form $e^{2\Phi_0} = -\frac{a^2}{l^2} + \left(1 - \frac{GM}{r}\right)^2$ and the rescaling interior metric is

$$ds^2 = -\left[-\frac{a^2}{l^2} + \left(1 - \frac{GM}{r}\right)^2\right] dt^2 + \frac{dr^2}{\left(1 - \frac{r_0^2}{r^2}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{45}$$

The exterior metric, $a \le r \le \infty$,, is same as the MTZ solution (14).

IV CONCLUSIONS

We have studied Morris-Thorne static traversable wormhole with a generic positive cosmological constant Λ by matching the internal and external geometries of two black solutions. In the internal region we impose a appropriate geometry to obtain a spherically symmetric traversable wormhole, while, in exterior region we use MTZ black hole solution. The surface tangential pressure with the surface energy density of the exotic matter is located at the throat of the wormhole. To match a vacuum exterior solution with interior solution, we have deduced an equation for the tangential surface pressure and another one which influences the behavior of the radial tension at the boundary.

We see that there is no wormhole solution with zero tangential pressure at $p = 0$, it form a boundary surface. The wormhole solutions are obtained with non-zero tangential pressure, i.e., $p \neq 0$. We briefly we represent some specific solutions of the traversable wormholes for different choices of the shape functions of the wormhole.

REFERENCES

1. M. S. Morris and K. S. Thorne, Am. J. Phys. **56,** 395 (1988).
2. Y. B. Zel'dovic Sov. Phys. Uspekhi, Vol. 11, No. 3, pp. **381** (1968).
3. S. Perlmutter et al., Nature (London) **391**, 51 (1998); Astrophysics. J. **517**, 565 (1999).
4. M. Heusler, Black Hole Uniqueness Theorems (Cambridge University Press, Cambridge, England, (1996).
5. V. Faraoni and S. Sonego, Class. Quant. Grav. **10**, 1185 (1993).
6. S. Kim, Phys. Lett. **A 166**, 13 (1992).

7. T. A. Roman, Phys. Rev. **D 47**, 1370 (1993).
8. M. S. R. Delgaty and R. B. Mann, Int. J. Mod. Phys. **D 4,** 231 (1995).
9. A. DeBenedicts and A. Das, Class. Quant. Grav. **18,** 1187 (2001).
10. A. DeBenedicts and A. Das, Nucl. Phys. Rev. **B 653,** 279 (2003).
11. S. V. Sushkov and Y. –Z. Zhang, Phys. Rev. **D 77,** 024042 (2008).
12. F. S. N. Lobo, Phys. Rev. **D 71,** 124022 (2005).
13. F. S. N. Lobo,Xiv:gr-qc/0410087.
14. F. S. N. Lobo, Class. Quant. Grav. **21,** 4811 (2004).
15. J. P. S. Lemos, F. S. N. Lobo and S. Q. de Oliveira, Phys. Rev. **D 68,** 064004 (2003).
16. F. S. N. Lobo and P. Crawford, Class. Quant. Grav. **21,** 391 (2004).
17. J. P. S. Lemos and F. S. N. Lobo, Phys. Rev. **D 69,** 104007 (2004).
18. J. P. S. Lemos, Phys. Lett. **B 352,** 46 (1995).
19. J. P. S. Lemos and V. T. Zanchin, Phys. Rev. **D 54,** 3840 (1996).
20. J. P. S. Lemos, arXiv:gr-qc/0011092.
21. J. P. S. Lemos and F. S. N. Lobo, Phys. Rev. **D 59,** 044020 (1999).